Indian Statistical Institute, Bangalore
B. Math (Hons.) Second Year
Second Semester - Ordinary Differential Equations
Semesteral Examination

Total marks: 60 **Maximum marks:** 50 **Date:** April 27, 2026
(Any score above 50 will be treated as 50)

Duration: 3 hours

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1. State whether the following statements are true or false, justify your answer.
   (a) The following differential equation can be solved with an integrating factor which is a function of xy: [3]
   $$(1 + xy - y^3)dx + (x^2 - xy^2 - 2y)dy = 0.$$
   (b) There exist continuous functions p, q on $[-1, 1]$ such that the function $y(t) = t^3$ is a solution of the second-order ODE
   $$y'' + p(t)y' + q(t)y = 0$$
   on the interval $(-1, 1)$. [3]
   (c) Any polynomial of degree $n \geq 1$ can be written as a finite linear combination of Legendre polynomials. [4]

2. Answer the following questions for the differential equation:
   $$x^2y''(x) + xy'(x) + (x^2 - 1)y(x) = 0, \ x > 0.$$
   (a) Show that there is a non-trivial analytic solution entered at 0. Obtain its power series, and show that it converges on $\mathbb{R}$. [5]
   (b) Prove that every solution has infinitely many zeros in $(0, \infty)$. [4]
   (c) Prove that every non-trivial solution has at most one zero in $(0, \frac{\pi}{4})$. [3]

3. Consider the two dimensional system:
   $$\dot{x} = y \quad \& \quad \dot{y} = -x + y(1 - x^2 - y^2).$$
   (a) Find all equilibrium points of the system. [1]
   (b) Rewrite the system in polar coordinates. Use this to classify the equilibrium points, and to conclude the existence of a closed orbit. [6]
   (c) Use the above information to draw the phase portrait of the system. [6]

4. Let $q : \mathbb{R} \to (0, \infty)$ be a continuous function such that $\lim_{x\to\infty} q(x) = 1$. Let $y : \mathbb{R} \to \mathbb{R}$ be a C^2 solution of $y''(x) + 2y'(x) = q(x)$. Prove the following:

(a) $\lim_{x\to\infty} y(x)/x = 1/2$. [5]

(b) y has finitely many zeros in $(0,\infty)$. [5]

(c) Find the general solution of the ODE when $q(x) = 1 + e^{-x}$. [5]

5. Let $f, g, G : \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the following conditions:

$$G(0) = g(0) = f(0) = 0, \quad gf > 0 \text{ and } G > 0 \text{ on } \mathbb{R} \setminus \{0\}, \quad G' = g.$$

Consider the following system:

$$\dot{x} = y - f(x) \quad \& \quad \dot{y} = -g(x).$$

Show that $(0,0)$ is an asymptotically stable equilibrium point by constructing a Lyapunov function in the following cases:

(a) $g(x) = x$ and $f(x) = x^3$. [4]

(b) For general f, g, G satisfying the above conditions. [6]

Good luck!!